Problem 4.28

Show that any open or closed interval in E^n is connected.

proof: Let $x \in I \subset E^n$, I is an open interval. For any point y in I, define the line segment between x and y as

$$S(x,y) = \{ \lambda x + (1-\lambda) y : 0 \le \lambda \le 1 \}. \subset I$$

Let $f : [0,1] \mapsto E^n$, $f(\lambda) = \lambda x + (1-\lambda) y = S(x,y)$

f is continuous in $[0,1] \Rightarrow f[0,1]$ is connected, i.e $\boxed{S_{xy} \text{ is connected.}}$

Then $\bigcup_{y \in I} S(x,y)$ is connected.

$\forall y \in I, \quad y \in S(x,y) \subset \bigcup_{y \in I} S(x,y)$

$\Rightarrow I \subset \bigcup_{y \in I} S(x,y)$.

Let $y \in I$, $\quad y \in (0,1) \times (0,1)$ and $x \in (0,1) \times (0,1)$

then $\quad \lambda x \in (0, \lambda) \times (0, \lambda)$

$\quad (1-\lambda) y \in (0, 1-\lambda) \times (0, 1-\lambda)$

$\Rightarrow \quad \lambda x + (1-\lambda) y \in (0,1) \times (0,1)$

i.e. $\quad \lambda x + (1-\lambda) y \in I, \quad \forall \lambda \in [0,1]$

then $\quad S(x,y) \subset I$

$\Rightarrow \quad \bigcup_{y \in I} S(x,y) \subset I.$

Thus, $\quad \bigcup_{y \in I} S(x,y) = I$

Problem 4.41

$f_k : E \mapsto \mathbb{R}$, $f = \lim_{k\to\infty} f_k$, E compact.

f and f_k are continuous.

$f_1(p) \le f_2(p) \le \cdots \le f_k(p) \le \cdots \quad \forall p \in E$.

Prove that $f_k \to f$ uniformly.

proof: Let $\varepsilon > 0$, $p_1, p_2 \in E$.

① $f = \lim_{k\to\infty} f_k \Rightarrow \exists K \in \mathbb{N}$, s.t. $k > K$

$\Rightarrow d(f(p), f_k(p)) < \frac{\varepsilon}{3}$.

② f, f_k continuous in E, E compact $\Rightarrow$

f, f_k are uniformly continuous in E. $\Leftrightarrow$

$\exists \delta_1 > 0$, s.t. $d(p_1, p_2) \Rightarrow d(f(p_1), f(p_2)) < \frac{\varepsilon}{3}$

$\exists \delta_2 > 0$, s.t. $d(p_1, p_2) \Rightarrow d(f_k(p_1), f_k(p_2)) < \frac{\varepsilon}{3}$

Choose $\delta = \min\{\delta_1, \delta_2\}$.

$d(f_k(p_1) - f(p_1)) \le d(f_k(p_1), f_k(p_2)) +$

$d(f_k(p_2), f(p_2)) + d(f(p_2), f(p_1))$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since $f_1(p) \le f_2(p) \le \cdots \le f_K(p) \le \cdots$

$d(f_K(p_1), f(p_1)) > d(f_K(p_0) - f(p_0))$, $\forall K \in \mathbb{N}$

Since E is compact, $E \subset \bigcup_{k=1}^{n} B_{\delta_k}(p_k)$.

Take $N = \max\{K(p_1), K(p_2), \cdots, K(p_n)\}$.

If $n > N$, $n > N_k$

Let $x \in E$, $x \in B_{\delta_k}(p_k)$ for some $1 \le k \le n$.

$\Rightarrow |f_n(x) - f(x)| < \varepsilon$ since $n \ge K(p_k)$.

Problem 5.9

$f: U \mapsto \mathbb{R}$, $g: U \mapsto \mathbb{R}$, f & g are differentiable in U. U open. $a \in U$.

$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

Prove that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the right limit exists.

proof: